1. Let $A = \{ r \in \mathbb{Q} \mid r > 0 \land r^2 > 3 \}$. Show that $A$ has a lower bound in $\mathbb{Q}$ but no greatest lower bound in $\mathbb{Q}$. Give all details of the proof along the lines of the proof given in the lecture that the rationals are not complete.

2. In addition to the completeness property, the Archimedean property is an important fundamental property of $\mathbb{R}$. It says that if $x, y \in \mathbb{R}$ and $x, y > 0$, there is an $n \in \mathbb{N}$ such that $nx > y$.

Use the Archimedean property to show that if $r, s \in \mathbb{R}$ and $r < s$, there is a $q \in \mathbb{Q}$ such that $r < q < s$. (Hint: pick $n \in \mathbb{N}$, $n > 1/(s - r)$, and find an $m \in \mathbb{N}$ such that $r < (m/n) < s$.)

3. Formulate both in symbols and in words what it means to say that $a_n \not\to a$ as $n \to \infty$.

4. Prove that $(n/(n + 1))^2 \to 1$ as $n \to \infty$.

5. Prove that $1/n^2 \to 0$ as $n \to \infty$.

6. Prove that $1/2^n \to 0$ as $n \to \infty$.

7. We say a sequence $\{a_n\}_{n=1}^\infty$ tends to infinity if, as $n$ increases, $a_n$ increases without bound. For instance, the sequence $\{n\}_{n=1}^\infty$ tends to infinity, as does the sequence $\{2^n\}_{n=1}^\infty$. Formulate a precise definition of this notion, and prove that both of these examples fulfil the definition.

8. Let $\{a_n\}_{n=1}^\infty$ be an increasing sequence (i.e. $a_n < a_{n+1}$ for each $n$). Suppose that $a_n \to a$ as $n \to \infty$. Prove that $a = \text{lub}\{a_n \mid n \in \mathbb{N}\}$.

9. Prove that if $\{a_n\}_{n=1}^\infty$ is increasing and bounded above, then it tends to a limit.