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## APPRECIATIVE REMARKS ON THE THEORY OF GROUPS.


By PROF. G. A. MILLER.


The teacher of mathematics is frequently called upon to make comments on various lines of mathematical thougnt. In doing so it is very desirable to be able to make use of remarks by prominent mathematicians. Some of these naturally exaggerate the relative importance of a particular subject and their true value can only be appreciated when compared with similar remarks on other mathematical developments. A number of such collections of appreciative remarks on modern subjects would doubtless prove very useful, especially to those who are in doubt as to which field to choose as an object of investigation.

In a recent German review of Professor Dickson's Linear Groups,* the reviewer aptly remarks that group theory is especially cultivated by the English speaking people. As there are very few subjects in pure mathematics in regard to which such a remark could be made by one so well informed as Professor Loewy, statements with reference to the relative importance of this subject may be of especial interest to the English speaking students of mathematics. It may be observed that most of the statements given below are dne to mathematicians whose native language is not English.

An important feature of group theory is exhibited by the new *International Catalogue of Scientific Literature*. Three subjects are classed under "Fundamental Notions;" viz., Bases of Arithmetic, General Algebra, and The Theory of Groups. As some of the leading mathematicians of the world helped to arrange this cat-

* Loewy, *Archiv der Mathematik und Physik*, Vol. 4, 1903, ~~page 333~~.

alogue, this classification is very significant. The three subjects with which group theory has most direct contact according to this catalogue are: Theory of Algebraic Equations, Automorphic Functions, and Differential Invariants.

That there are many other subjects of direct contact may be seen from the following quotations: "The theory of congruences bases itself substantially upon the fundamental concept of mathematics, which is already the foundation of Poinsot's method, the concept of group."* "A large part of the theory of numbers is only the theory of abelian groups."† "It might be said of the most important parts of recent geometry that one conception dominates everywhere; that is the conception of the group."‡ "The most important of all these view points is furnished by the theory of groups, which is really a creation of our century and has shown its dominating influence in nearly all parts of mathematics; not only in the recent theories but also far towards the foundation of the subject, so that this theory can no longer be omitted in the elementary text-books."§

"There are two things which have become especially important for the latest development of algebra; that is, on the one hand the ever more dominating theory of groups whose systematizing and clarifying influence can be felt everywhere, and then the deep penetration of number theory."‖ "The concepts, group and invariant, take each day a more preponderant place in mathematics and tend to dominate this entire science."¶ "The mathematics of the twenty-first century may be very different from our own; perhaps the schoolboy will begin algebra with the theory of substitution-groups, as he might now but for inherited habits."**

"In fine, the principal foundation of Euclid's demonstrations is really the existence of the group and its properties. Unquestionably he appeals to other axioms which it is more difficult to refer to the notion of group. An axiom of this kind is that which some geometers employ when they define a straight line as the shortest distance between two points. But it is precisely such axioms that Euclid enunciates. The others, which are more directly associated with the idea of displacement and with the idea of groups are the very ones which he implicitly admits and which he does not deem even necessary to enunciate. This is tantamount to saying that the former are the fruit of later experience, that the others were first assimilated by us, and that consequently the notion of group existed prior to all others."††

"Although we can not give here a complete exposition of the fundamental results of Sophus Lie in the theory of continuous transformation groups, yet it is indispensable that we make some general remarks on this notion of continuous

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* Bachmann, *Die Elemente der Zahlentheorie*, Vol. 1, 1892, Preface.

† Frobenius, *Sitzungsberichte*, 1893, page 627.

‡ Maschke, THE AMERICAN MATHEMATICAL MONTHLY. Vol. 9, 1902, page 214.

§ Pund, *Algebra mit Einschluss dei elementaren Zahlentheorie*, 1899, Preface.

‖ Weber, *Lehrbuch der Algebra*, Vol. 1, 1898, Preface

¶ Lie, *Le Centenaire de l' Ecole Normale*, 1895, page 485.

** Newcomb, *Bulletin of the New York Mathematical Society*, Vol. 3, 1893, page 107.

†† Poincare, *The Monist*, Vol. 9, 1898, page 34. On page 41 of the same article Poincare says, "What we call geometry is nothing but the study of formal properties of a certain continuous group; so that we may say, space is a group."

groups which play such an important rôle in the science of our epoch."* "The group concept was employed in the preceding century (about 1770) simultaneously by Lagrange and Vandermonde, and since this time it occupies a prominent place in the theory of algebraic equations. In regard to this it is only necessary to refer to the name of Galois. Hence group theory has been regarded as a supplement of algebra. This however is incorrect. For the group concept extends far beyond this into almost all other parts of mathematics."†

"It was reserved for Galois to place the theory of equations on a definite foundation by showing that to each equation there corresponds a substitution group in which are exhibited its essential characteristics, and especially those which relate to its solution by other auxiliary equations."‡ "I should reproach myself for forgetting, even in so rapid a resumé, the applications which Lie has made of his theory of groups to the non-Euclidean geometry and to the profound study of the axioms which lie at the basis of our geometric knowledge."§

"The theory of groups, which is making itself felt in nearly every part of higher mathematics, occupies the foremost place among the auxiliary theories which are employed in the most recent function theory."‖ "It need scarcely be added that some modern mathematicians seem to avoid group theory even where it would simplify the treatment of the subject in hand. This seems to be true, for instance, of Hilbert's Grundlagen der Geometrie."¶

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## OUR SYMBOL FOR ZERO.

By DR. GEORGE BRUCE HALSTED.

At the Paris International Congress, the paper of my erudite friend of Japanese days, Professor Fiyisawa, attracted, I believe, apart from the great address of Hilbert, more favorable attention than any other.

In praising this paper, I ventured there to emphasize, that the appearance in Japan, before any communication with Europe, of a positional notation for number with precisely the symbol for zero which we now use, and which, as Professor Cajori says in the February number of the AMERICAN MATHEMATICAL MONTHLY, has been supposed of Hindu origin, raised the question for the future historian of mathematics of the relation or connection between these two indubitable, however widely separated, appearances of the same peculiar symbol.

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* Picard, *Traite d' analyse*, Vol. 3, 1896, page 492.

† Klein, *Einleitung in die hoehere Geometrie* II, 1893, page 3.

‡ Jordan, *Traite des substitutions*, 1870, Preface.

§ Darboux, *Comptes Rendus*, Vol. 128, 1899, page 528.

‖ Fricke und Klein, *Automorphe Functionen*, Vol. 1, 1897, page 1.

¶ Poincare, *Bulletin des Sciences Mathematiques*, Vol. 26, 1902, page 272.